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Semidiscrete Least-Squares Methods for Second Order Parabolic Problems With Nonhomogenous Data

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Abstract. Recently, Bramble and Thomée proposed semidiscrete least-squares methods for the heat equation. In this paper we extend these methods to variable coefficient parabolic operators with nonhomogeneous equations and boundary conditions.

1. Introduction. Recently there has been much interest in variational methods for approximating the solution of parabolic problems (cf. [8], [12], [14]).

For essential boundary conditions, the author proposed weighted least-squares methods for parabolic problems [10]. These methods are based on the ideas of Bramble and Schatz [3]. Another way of implementing the ideas of [3] to parabolic problems has been proposed by Bramble and Thomée [6]. In [6] the authors give semidiscrete least-squares methods for the heat equation under homogeneous boundary conditions.

The purpose of this paper is to generalize the methods of [6] to variable coefficient second order parabolic operators with nonhomogeneous data. The error analysis given here will be similar to that of [6], however we need to significantly change some details.

2. The Initial-Boundary Value Problem. Let Ω be a bounded open domain in Euclidean N-space with boundary $\partial\Omega$, of class C^∞ (cf. [1]). For $0 < T < \infty$ we put $Q = \Omega \times (0, T]$ and $\Gamma = \partial\Omega \times (0, T]$.

We shall consider, in Q, the parabolic operator L defined by

$$Lu = A(x, t)u - u_t = \sum_{i,j=1}^{N} D_{x_i}(a_{ij}(x, t) D_{x_j} u) - u_t$$

and the following initial-boundary value problem:

$$Lu = f \quad \text{in } Q, \qquad u = g \quad \text{on } \Gamma, \qquad u(\cdot, 0) = u_0 \quad \text{in } \Omega \tag{2.1}$$

where f, g, u_0 are given. We assume that A is uniformly elliptic in the closure of Q, $\bar{Q}$, and for convenience assume that the coefficients a_{ij} are in $C^\infty(\bar{Q})$.

We will need the following function spaces. For $k \geqq 0$, an integer, $H^k(\Omega)$ will denote the usual Sobolev space of order k on Ω with norm

$$\|\phi\|_k = \left[\sum_{|\alpha| \leq k} \|D^\alpha \phi\|^2\right]^{1/2} \tag{2.2}$$

where $\|\phi\| = (\int_\Omega |\phi(x)|^2 \, dx)^{1/2}$.

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We will denote the corresponding inner product on $H^0(\Omega) = L_2(\Omega)$ by $\langle\langle \cdot, \cdot \rangle\rangle$. The inner product and norm on $L_2(\partial\Omega)$ will be denoted by $\langle \cdot, \cdot \rangle$ and $|\cdot|$ respectively.

For the semidiscrete methods which follow we will need to introduce finite-dimensional subspaces of $H^2(\Omega)$ having certain approximation theoretic properties depending on a small parameter h. Specifically we will assume throughout this paper that S_h^r is any finite-dimensional subspace of $H^2(\Omega)$ having the following property:

(*) For any $u \in H^s(\Omega)$ with $2 \leqq s \leqq r$ there exists a constant C, independent of h and u, such that

$$\inf_{\phi \in S_h^r} \sum_{j=0}^{2} h^{j-2} \|u - \phi\|_j \leqq C h^{s-2} \|u\|_s .$$

Examples of such subspaces are contained in [2], [7], [9], and [13].

3. Implicit Semidiscrete Methods. In this section we give semidiscrete least-squares schemes for finding approximate solutions of the initial-boundary value problem using the subspaces S_h^r. The schemes proposed here are based on the classical implicit finite-difference approximation for L.

Before making these ideas more precise we will need some notation. Let $k = T/M$ where M is a positive integer and let $t_n = nk$ for $n = 0, \cdots, M$. We will use the notation $u^n = u(\cdot, t_n)$ and $A_n = A(\cdot, t_n)$.

We shall approximate the solution of problem (2.1) by a function of the form

$$v(x, t) = \sum_{j=1}^{\nu} c_j(t) \phi_j(x) \tag{3.1}$$

where $\{\phi_j\}_{j=1}^{\nu}$ is a suitable local basis for S_h^r and c_j is defined on the mesh: $0 < t_1 < \cdots < t_{M-1} < T$.

The implicit approximation schemes are defined as follows:

For given S_h^r having local basis $\{\phi_j\}_{j=1}^{\nu}$ and given γ satisfying $0 \leqq \gamma \leqq \frac{3}{2}$, find a function v of the form (3.1) such that

$$\begin{aligned}
&\text{(i)} \quad \langle\langle u_0 - v^0, \phi \rangle\rangle = 0 \quad \text{for all } \phi \in S_h^r \quad \text{and for } n = 0, \cdots, M-1, \\
&\text{(ii)} \quad \langle\langle k f^{n+1} - v^n - (kA_{n+1} - 1)v^{n+1}, (kA_{n+1} - 1)\phi \rangle\rangle \\
&\qquad + k^2 h^{-2\gamma} \langle g^{n+1} - v^{n+1}, \phi \rangle = 0 \quad \text{for all } \phi \in S_h^r.
\end{aligned} \tag{3.2}$$

We remark that the elements of S_h^r are not required to satisfy boundary conditions, and only L_2 inner products are involved in the computation of v. The solution of (3.2), for each n, is determined by solving a linear system of algebraic equations whose coefficients depend only on $F = \{f, g, u_0\}$, h, k, and γ. The linear system yields a symmetric matrix which is sparse if we take S_h^r to be a certain class of splines with a suitable local basis. Note also that we could choose $v^0 \equiv u_0$ in place of (i) in (3.2).

For notational simplicity it is convenient to introduce the following definitions. For $u \in H^2(\Omega)$ we define the operator $L_k^n u \equiv (kA_n - 1)u$.

Let Λ be the product space $H^0(\Omega) \times H^0(\partial\Omega)$ with inner product

$$[U, V] = \langle\langle f, \tilde{f} \rangle\rangle + k^2 h^{-2\gamma} \langle g, \tilde{g} \rangle$$

where $U = \{f, g\}$ and $V = \{\tilde{f}, \tilde{g}\}$. We put $\|U\|_\Lambda = [U, U]^{1/2}$.

Let T_k^n be the map $\{L_k^n u, u|_{\partial\Omega}\}$ from $H^2(\Omega)$ into Λ. Then $\|T_k^n u\|_\Lambda$ is a norm on $H^2(\Omega)$.

Existence. We assume that $u_0 \in H^0(\Omega)$, $f^n \in H^0(\Omega)$, and $g^n \in H^0(\partial\Omega)$ for each $n = 1, \cdots, M$. Since the matrix $\{\langle\langle\phi_i, \phi_j\rangle\rangle\}$ is positive definite, (i) has a unique solution $\{c_1^0, \cdots, c_\nu^0\}$. Now suppose that $\{c_1^n, \cdots, c_\nu^n\}$ have been found; then (ii) has a unique solution if and only if the only solution of the system

$$[T_k^{n+1}v^{n+1}, T_k^{n+1}\phi_j] = 0, \qquad 1 \leqq j \leqq \nu, \tag{3.3}$$

is the trivial solution. Multiplying (3.3) by c_j^{n+1} and summing over j yields

$$\|T_k^{n+1}v^{n+1}\|_\Lambda^2 = 0$$

so that $L_k^{n+1}v^{n+1} = 0$ in Ω and $v^{n+1} = 0$ on $\partial\Omega$. Hence $v^{n+1} \equiv 0$.

Error Analysis. We will need the following a priori inequality for the operator L_k^n which is essentially proved in [6].

LEMMA 3.1. *There exists a constant* $\alpha \geqq 1$ *such that, for* $u \in H^2(\Omega)$ *and* $1 \leqq n \leqq M$,

$$\|u\|^2 \leqq \|L_k^n u\|^2 + \alpha k^{1/2}|u|^2. \tag{3.4}$$

We will also make use of the following lemma which is essentially proved in [6].

LEMMA 3.2. *Suppose* $u \in H^s(\Omega)$ *with* $2 \leqq s \leqq r$ *and* $C_0 > 0$. *If* $k \geqq C_0 h^{4\gamma/3}$ $(0 \leqq \gamma \leqq \frac{3}{2})$ *then*

$$\inf_{\phi \in S_h^r} \|T_k^n(u - \phi)\|_\Lambda \leqq Ckh^{s-2}\|u\|_s \tag{3.5}$$

where C is independent of h, k, and u.

We will also make use of the following easily proved estimate.

LEMMA 3.3. *Suppose* a_n, b_n, *and* c_n *are nonnegative. If* $\sigma \geqq 0$, *and, for* $n = 0, 1, \cdots, M-1$,

$$a_{n+1}^2 \leqq b_{n+1}^2 + \left(1 + \frac{\sigma}{M}\right)a_n^2 + c_{n+1}^2 + 2\left(1 + \frac{\sigma}{M}\right)^{1/2} a_n c_{n+1}$$

then there exists a constant C, independent of M, such that

$$\max_{0 \leqq n \leqq M} a_n \leqq C\left\{M^{1/2} \max_{0 \leqq n \leqq M} b_n + M \max_{0 \leqq n \leqq M} c_n + a_0\right\}.$$

We are now in a position to analyze the error in scheme (3.2). Let u be the solution of (2.1), v be the solution of (3.2), and put $e = u - v$.

THEOREM 3.1. *If* $u(\cdot, t) \in H^s(\Omega)$ *for each* $t \in (0, T]$ *with* $2 \leqq s \leqq r$ *and* $u \in C^2(\bar{Q})$, *then, for* $k \geqq \alpha^{2/3}h^{4\gamma/3}$, *we have*

$$\max_{0 \leqq n \leqq M} \|e^n\| \leqq C\left\{h^2\|u_0\|_2 + k^{1/2}h^{s-2} \max_{1 \leqq n \leqq M} \|u^n\|_s + k \max_{t \in [0,T]} \|D_t^2 u(\cdot, t)\|\right\}. \tag{3.6}$$

We remark that α is the constant of Lemma 3.1.

Proof. We have from (3.4) that

$$\|e^{n+1}\|^2 \leqq \|L_k^{n+1}e^{n+1}\|^2 + \alpha k^{1/2}|e^{n+1}|^2$$

so that, for $k \geqq \alpha^{2/3}h^{4\gamma/3}$,

$$\|e^{n+1}\|^2 \leqq \|T_k^{n+1}e^{n+1}\|_\Lambda^2 = [T_k^{n+1}e^{n+1}, T_k^{n+1}u^{n+1}] - [T_k^{n+1}e^{n+1}, T_k^{n+1}v^{n+1}]. \tag{3.7}$$

Now let $\psi^{n+1} \in S_h^r$ be such that

$$[T_k^{n+1}(u^{n+1} - \psi^{n+1}), T_k^{n+1}\phi] = 0 \quad \text{for all } \phi \in S_h^r. \tag{3.8}$$

By an elementary application of Taylor's theorem we have

(3.9) $$L_k^{n+1}u^{n+1} = kf^{n+1} - u^n + \rho^{n+1}$$

where $\|\rho^{n+1}\| \leqq Ck^2 \max_{t\in[0,T]}\|D_t^2u(\cdot\,,t)\|$.

We now use (3.2) and (3.9) in (3.7) to yield

(3.10) $$\|e^{n+1}\|^2 \leqq \langle\langle e^n - \rho^{n+1}, L_k^{n+1}(v^{n+1} - \psi^{n+1})\rangle\rangle + [T_k^{n+1}e^{n+1}, T_k^{n+1}(u^{n+1} - \psi^{n+1})].$$

By virtue of (3.10) and the definition of ψ^{n+1} we then have

(3.11) $$\|e^{n+1}\|^2 \leqq \langle\langle e^n - \rho^{n+1}, L_k^{n+1}(v^{n+1} - \psi^{n+1})\rangle\rangle + \|T_k^{n+1}(u^{n+1} - \psi^{n+1})\|_\Lambda^2.$$

We now estimate the first term on the right-hand side of (3.11). We subtract (3.8) from (3.2) and use the relation (3.9) to yield

(3.12) $$[T_k^{n+1}(v^{n+1} - \psi^{n+1}), T_k^{n+1}\phi] = \langle\langle e^n - \rho^{n+1}, L_k^{n+1}\phi\rangle\rangle$$

where $\phi \in S_h^r$ is arbitrary. Choosing $\phi = v^{n+1} - \psi^{n+1}$ in (3.12) implies that

$$\|L_k^{n+1}(v^{n+1} - \psi^{n+1})\|^2 \leqq |\langle\langle e^n - \rho^{n+1}, L_k^{n+1}(v^{n+1} - \psi^{n+1})\rangle\rangle|$$

so that, by the Schwarz inequality,

(3.13) $$\|L_k^{n+1}(v^{n+1} - \psi^{n+1})\| \leqq \|e^n - \rho^{n+1}\|.$$

Using the Schwarz inequality and (3.13) in (3.11) yields

(3.14) $$\|e^{n+1}\|^2 \leqq \|T_k^{n+1}(u^{n+1} - \psi^{n+1})\|_\Lambda^2 + \|e^n - \rho^{n+1}\|^2$$

so that

(3.15) $$\|e^{n+1}\|^2 \leqq \|T_k^{n+1}(u^{n+1} - \psi^{n+1})\|_\Lambda^2 + \|e^n\|^2 + \|\rho^{n+1}\|^2 + 2\,\|e^n\|\,\|\rho^{n-1}\|.$$

It follows from (3.15) by an application of Lemma 3.3 with $\sigma = 0$ that

(3.16) $$\max_{1\leqq n\leqq M}\|e^n\| \leqq C\Big\{k^{-1/2}\max_{1\leqq n\leqq M}\|T_k^n(u^n - \psi^n)\|_\Lambda + \|u_0 - v^0\| + k^{-1}\max_{1\leqq n\leqq M}\|\rho^n\|\Big\}.$$

The result (3.6) follows from (3.16) by an application of Lemma 3.2 and property (*).

4. Crank-Nicolson Semidiscrete Methods. The implicit schemes of the previous section have, at best, order of convergence $O(k)$. In order to improve the order of convergence it is necessary to discretize the time variable with higher order accuracy. The schemes given here are based on a Crank-Nicolson type finite-difference approximation for the operator L. It will be shown that the order of convergence can be improved to $O(k^2)$.

We will use the notation $u^{n+1/2} = \frac{1}{2}(u^n + u^{n+1})$. We shall again approximate the solution of problem (2.1) by a function of the form (3.1); however, we will now take $v^0 \equiv u_0$. We will use the notation $\tilde{L}_k^n u = (kA_n + 1)u$.

The approximation schemes are defined as follows:

For a given S_h^r having local basis $\{\phi_i\}_{i=1}^{\nu}$ and given γ satisfying $0 \leqq \gamma \leqq \frac{3}{2}$, find a function of the form (3.1), with $u_0 \equiv v^0$, such that, for $n = 0, \cdots, M-1$,

$$(4.1) \qquad \langle\langle kf^{n+1/2} - \tilde{L}^n_{k/2}v^n - L^{n+1}_{k/2}v^{n+1}, L^{n+1}_{k/2}\phi\rangle\rangle + k^2h^{-2\gamma}\langle g^{n+1} - v^{n+1}, \phi\rangle = 0$$

for all $\phi \in S_h^r$.

Error Analysis. We will again need an a priori inequality for the operator $L_{k/2}^n$ which is essentially proved in [6].

LEMMA 4.1. *There is a constant $\beta \geqq 1$ such that, for $u \in H^2(\Omega)$ and $1 \leqq n \leqq M$,*

$$(4.2) \qquad ||\tilde{L}^n_{k/2}u||^2 \leqq (1 + k)\{||L^n_{k/2}u||^2 + \beta|u|^2\}.$$

Let u be the solution of (2.1), v be the solution of the Crank-Nicolson scheme (4.1), and put $e = u - v$.

THEOREM 4.1. *If $u(\cdot\,, t) \in H^s(\Omega)$ for each $t \in (0, T]$ with $2 \leqq s \leqq r$ and $u \in C^3(\bar{Q})$ then for $k \geqq C_0h^\gamma$, with $C_0 \geqq \max\{\alpha^{2/3}, \beta\}$ we have*

$$(4.3) \qquad \max_{1\leqq n\leqq M} ||e^n|| \leqq C\Big\{k^{1/2}h^{s-2} \max_{1\leqq n\leqq M} ||u^n||_s + k^2 \max_{t\in[0,T]} ||D_t^3u(\cdot\,, t)||\Big\}$$

where α and β are the constants of Lemmas 3.1 *and* 4.1 *respectively and C is independent of h, k, and u.*

Note that we have assumed the more restrictive condition $k \geqq C_0h^\gamma$.

Proof. As in the proof of Theorem 3.1 we have

$$(4.4) \qquad ||e^{n+1}||^2 \leqq ||T^{n+1}_{k/2}e^{n+1}||^2_\Lambda.$$

Using the relation

$$(4.5) \qquad L^{n+1}_{k/2}u^{n+1} = kf^{n+1/2} - \tilde{L}^n_{k/2}u^n + \rho^{n+1/2}$$

where

$$||\rho^{n+1/2}|| \leqq Ck^3 \max_{t\in[0,T]} ||D_t^3u(\cdot\,, t)||$$

and arguments analogous to those in the proof of Theorem 3.1 it can be shown that the right-hand side of (4.4) is bounded by

$$(4.6) \qquad ||T^{n+1}_{k/2}(u^{n+1} - \psi^{n+1})||^2 + |\langle\langle\tilde{L}^n_{k/2}e^n - \rho^{n+1/2}, L^{n+1}_{k/2}(v^{n+1} - \psi^{n+1})\rangle\rangle|$$

where $\psi^{n+1} \in S_h^r$ is defined such that

$$(4.7) \qquad [T^{n+1}_{k/2}(u^{n+1} - \psi^{n+1}), T^{n+1}_{k/2}\phi] = 0 \quad \text{for all } \phi \in S_h^r.$$

We estimate the last term in (4.6) by methods analogous to those used in the proof of Theorem 3.1. We subtract (4.7) from (4.1) and use the relation (4.5) to yield

$$(4.8) \qquad [T^{n+1}_{k/2}(v^{n+1} - \psi^{n+1}), T^{n+1}_{k/2}\phi] = \langle\langle\tilde{L}^n_{k/2}e^n - \rho^{n+1/2}, L^{n+1}_{k/2}\phi\rangle\rangle$$

where $\phi \in S_h^r$ is arbitrary. Choosing $\phi = v^{n+1} - \psi^{n+1}$ in (4.8) implies that

$$(4.9) \qquad ||L^{n+1}_{k/2}(v^{n+1} - \psi^{n+1})|| \leqq ||\tilde{L}^n_{k/2}e^n - \rho^{n+1/2}||.$$

Hence, from (4.4) through (4.9), we conclude that, for $0 \leqq n \leqq M - 1$,

$$(4.10) \qquad ||e^{n+1}||^2 \leqq ||T^{n+1}_{k/2}e^{n+1}||^2_\Lambda \leqq ||T^{n+1}_{k/2}(u^{n+1} - \psi^{n+1})||^2_\Lambda + ||\tilde{L}^n_{k/2}e^n - \rho^{n+1/2}||^2.$$

Hence from the assumption $k \geqq C_0h^\gamma$ and Lemma 4.1 it follows that

$$||e^{n+1}||^2 \leqq ||T_{k/2}^{n+1}e^{n+1}||_\Lambda^2 \leqq ||T_{k/2}^{n+1}(u^{n+1} - \psi^{n+1})||_\Lambda^2$$

$$+ (1 + k)\,||T_{k/2}^n e^n||_\Lambda^2 + ||\rho^{n+1/2}||^2 + 2(1 + k)^{1/2}||\rho^{n+1/2}||\,||T_{k/2}^n e^n||_\Lambda. \tag{4.11}$$

An application of Lemma 3.3 with $\sigma = T$ in (4.11) yields

$$\max_{1 \leqq n \leqq M} ||T_{k/2}^n e^n||_\Lambda \leqq C\left\{k^{-1/2} \max_{1 \leqq n \leqq M} ||T_{k/2}^n(u^n - \psi^n)||_\Lambda + k^{-1} \max_{1 \leqq n \leqq M} ||\rho^{n-1/2}||\right\} \tag{4.12}$$

from which (4.3) easily follows by an application of Lemma 3.2 and observing that (4.4) is valid for any $n = 0, \cdots, M - 1$.

Discussion of Results. The bounds in the error estimates (3.6) and (4.3) involve certain norms of the solution u of problem (2.1). Under slightly stronger regularity assumptions on u (i.e., the data $F = \{f, g, u_0\}$ must be sufficiently regular) these norms on u can be bounded by certain Hölder norms on the data (cf. Theorem 5.2 on p. 320 of [11]).

For the purely implicit scheme (3.2) suppose we take $r = 4$ and S_h^4 to be cubic splines or the Hermite space of piecewise cubic polynomials ([4], [5], [9]). Then if $||u(\cdot\,, t)||_s$ is bounded uniformly in t, with $2 \leqq s \leqq r$, we have

$$\max_{0 \leqq n \leqq M} ||e^n|| = O(k^{1/2}h^{s-2} + k).$$

Hence, since $k \geqq \alpha^{2/3}h^{4\gamma/3}$, we have that the error is of order $O(k)$ for $s = 4$ and $\gamma = \frac{3}{2}$ or for $s = 3$ and $\gamma = \frac{3}{2}$.

For the Crank-Nicolson scheme (4.1) we find that the error is $O(k^2)$ if $\gamma = \frac{4}{3}$ and $||u(\cdot\,, t)||_4$ is bounded uniformly in t.

The Crank-Nicolson methods of [8] yield error estimates of the form

$$\max_{0 \leqq n \leqq M} ||e^n|| = O(h^{2s-1} + k^2)$$

where the solution of (2.1) satisfies $u(\cdot\,, t) \in H^{2s}$ and the space of approximants consists of piecewise polynomials of degree $2s - 1$ (on a mesh of width h). Thus if $s = 2$ (i.e., piecewise cubics) then the error is of order $O(k^2)$ provided $k \geqq Ch^{3/2}$. We should remark that the methods of [8] require the approximants to satisfy boundary conditions.

The methods of [10] yield error estimates of the form

$$\left\{\int_0^T ||e(\cdot\,, t)||^2\, dt\right\}^{1/2} = O(h^{2s} + k^s)$$

where $u \in H^{2s,s}(Q)$ (cf. [10]) and the admissible approximants consist of a certain class of splines which are piecewise polynomials of degree $2s - 1$ in the space variables (on a mesh of width h) and of degree $s - 1$ in t (on a mesh of width k). Thus for $s = 2$ and $k \geqq Ch^2$ the error is of order $O(k^2)$. We remark that the methods of [10] yield arbitrarily high order of accuracy (for sufficiently regular u) without requiring the admissible approximants to satisfy boundary conditions.

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